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Unsteady laminar boundary layers in a compressible stagnation flow

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The unsteady laminar compressible boundary-layer flow in the immediate vicinity of a two-dimensional stagnation point due to an incident stream whose velocity varies arbitrarily with time is considered. The governing partial differential equations, involving both time and the independent similarity variable, are transformed into new co-ordinates with finite ranges by means of a transformation which maps an infinite interval into a finite one. The resulting equations are solved by converting them into a matrix equation through the application of implicit finite-difference formulae. Computations have been carried out for two particular unsteady free-stream velocity distributions: (i) a constantly accelerating stream and (ii) a fluctuating stream. The results show that in the former case both the skin-friction and the heat-transfer parameter increase steadily with time after a certain instant, while in the latter they oscillate, thus responding to the fluctuations in the free-stream velocity.

1. Introduction

Many current aerodynamic problems involve accelerated or decelerated rocket missiles, blades rotating in non-uniform air streams, unsteady nozzle flow or oscillating wings, etc. In order to determine the friction drag and the rate of heat transfer through the surface for such unsteady motions, the nature of time-dependent laminar compressible boundary-layer flow of a fluid past an obstacle must be investigated. Attempts to obtain practical solutions, exact or approximate, to the complete set of boundary-layer equations resulting from the introduction of time into the analysis lead to very great difficulties. Consequently, most of the investigations of unsteady laminar boundary layers have been concerned with either bodies of simple shape, specific time variations of the velocity in the undisturbed flow or both. For instance, Lighthill (1954) has considered the case of an arbitrary cylinder fixed in a stream which fluctuates in magnitude but not in direction. Nevertheless, Yang (1958, 1959) has studied the development of the unsteady two-dimensional incompressible laminar boundary layer on an arbitrary cylinder fixed in a stream whose velocity varies arbitrarily with time by adopting an integral procedure. On the other hand, for compressible flows, only particular cases of this general problem have been considered (Moore 1951; Ostrach 1955; Moore & Ostrach 1956; Illingworth 1958; Gribben 1961, 1971),

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through the use of integral methods or series expansion methods. Both methods have their own drawbacks. Series expansion methods are somewhat laborious to apply and require many terms for large times. Integral methods are only approximate, not exact.

However, for unsteady two-dimensional laminar compressible boundary-layer flows, a suitable similarity transformation can be used to reduce the number of independent variables from three to two. The solutions of the resulting partial differential equations, involving only time and the independent similarity variable, are called semi-similar solutions. These semi-similar solutions may be obtained through the application of the finite-difference numerical technique. But, when finite-difference methods are employed for the numerical solution of problems concerned with regions of infinite extent in one or more directions, there are two main difficulties to be faced, viz. the satisfaction of the boundary conditions at infinity and the representation of an infinite interval with a finite number of grid points.

Of the three approaches (Sills 1969) that are commonly used to overcome these difficulties, the one that is most suited for the application of finite-difference schemes is to map the infinite field into a finite one by introducing a co-ordinate transformation. Sills (1969) has presented three transformations which map one or both of the intervals $(0,\infty)$ and $(-\infty,\infty)$ into finite intervals along with an illustration of their application to flow problems. The asymptotic nature of these transformations allows the straightforward application of the boundary conditions at infinity, at the same time concentrating the grid points in the desired region and eliminating the necessity of searching for the effective boundary-layer edge through additional iteration. Marvin & Sheaffer (1969) have used one of these transformations to solve non-similar compressible boundary-layer equations including foreign-gas injection. El Assar (1970) has treated the problem of the compressible laminar wake behind a thin flat plate by employing the above technique.

The present paper deals with the unsteady laminar compressible boundary-layer flow in the immediate vicinity of a two-dimensional stagnation point, the unsteadiness arising from arbitrary variations in the velocity of the incident stream with time. The equations governing the flow are transformed into new co-ordinates with a finite domain using a transformation given by Sills (1969). They are then solved numerically by converting them into a matrix equation with the help of implicit finite-difference formulae. Computations have been carried out for the flow past a cylinder for the following free-stream velocity distributions: (i) a stream moving with constant acceleration and (ii) a stream fluctuating about a steady mean. It may be noted that here the fluctuating-flow problem is solved as an initial-value problem starting with a steady solution and hence the solution is not the same as in other work (Lighthill 1954; Yang 1958; Gribben 1961, 1971) where transient motions are assumed to have died away.

2. Basic equations

The equations governing the unsteady two-dimensional laminar compressible boundary-layer flow of a fluid past an obstacle are the following:

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v) = 0, \tag{2.1a}$$

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right) = -\frac{\partial p_e}{\partial x}+\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right), \tag{2.1b}$$

$$\rho\left(\frac{\partial h}{\partial t}+u\frac{\partial h}{\partial x}+v\frac{\partial h}{\partial y}\right)-\left(\frac{\partial p_e}{\partial t}+u\frac{\partial p_e}{\partial x}\right) = \frac{1}{\sigma}\frac{\partial}{\partial y}\left(\mu\frac{\partial h}{\partial y}\right)+\mu\left(\frac{\partial u}{\partial y}\right)^2. \tag{2.1c}$$

Here t is the time, x and y are curvilinear co-ordinates along and perpendicular to the boundary, u and v the corresponding velocity components, p, ρ, μ and h the pressure, density, viscosity and specific enthalpy respectively and σ the Prandtl number, supposed to be constant. The suffix e refers to free-stream values.

The pertinent initial and boundary conditions are the following:

$$u(x,y,0) = u_i(x,y), \quad v(x,y,0) = v_i(x,y), \quad h(x,y,0) = h_i(x,y); \tag{2.2a}$$

$$u(x,0,t) = 0, \quad v(x,0,t) = v_w(x,t), \quad h(x,0,t) = h_w(x,t); \tag{2.2b}$$

$$u(x,\infty,t) = u_e(x,t), \quad h(x,\infty,t) = h_e(x,t). \tag{2.2c}$$

The suffixes i and w are used to denote values at the initial time $t = 0$ and at the wall $y = 0$ respectively. The free-stream velocity u_e, density ρ_e and enthalpy h_e, which are all functions of x and t in general, obey the equations

$$\partial\rho_e/\partial t+\partial(\rho_e u_e)/\partial x = 0, \tag{2.3a}$$

$$\rho_e\left(\frac{\partial u_e}{\partial t}+u_e\frac{\partial u_e}{\partial x}\right) = -\frac{\partial p_e}{\partial x}, \tag{2.3b}$$

$$\rho_e\left(\frac{\partial h_e}{\partial t}+u_e\frac{\partial h_e}{\partial x}\right) = \left(\frac{\partial p_e}{\partial t}+u_e\frac{\partial p_e}{\partial x}\right). \tag{2.3c}$$

Suppose that the fluid is a perfect gas and the external stream is homentropic. Also, assume for the sake of simplicity that the density–viscosity product $\rho\mu$ is directly proportional to $\rho_w\mu_w$, and introduce new variables ψ and Y defined by (Moore 1951; Illingworth 1958; Gribben 1961, 1971)

$$\left.\begin{aligned} Y &= \frac{1}{\rho_w}\int_0^y \rho\,dy, \\ u &= \frac{\rho_w}{\rho}\frac{\partial\psi}{\partial y} = \frac{\partial\psi}{\partial Y}, \quad v = -\frac{\rho_w}{\rho}\left(\frac{\partial\psi}{\partial x}+\frac{\partial Y}{\partial t}\right), \end{aligned}\right\} \tag{2.4}$$

so that the equation of continuity (2.1*a*) is satisfied identically. Then the momentum and energy equations, (2.1*b*) and (2.1*c*) respectively, become, for small Mach numbers (in which case the dissipation term may be neglected),

$$\left(\frac{\partial}{\partial t}+\frac{\partial\psi}{\partial Y}\frac{\partial}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial}{\partial Y}\right)\frac{\partial\psi}{\partial Y} = \left(\frac{\partial u_e}{\partial t}+u_e\frac{\partial u_e}{\partial x}\right)\frac{h}{h_e}+\chi\nu_w\frac{\partial^3\psi}{\partial Y^3}, \tag{2.5a}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial\psi}{\partial Y}\frac{\partial}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial}{\partial Y}\right)\frac{h}{h_e} = \frac{\chi\nu_w}{\sigma}\frac{\partial^2}{\partial Y^2}\left(\frac{h}{h_e}\right), \tag{2.5b}$$

where

$$\chi = \rho\mu/\rho_w\mu_w. \tag{2.5c}$$

It is the aim of the present investigation to calculate the critical parameters of the laminar compressible boundary-layer flow in the vicinity of a blunt-nosed cylinder when the velocity of the upstream undisturbed flow varies arbitrarily with time. For two-dimensional stagnation-point flow, without loss of generality, the free-stream velocity distribution u_e can be taken to be of the form (Yang 1958)

$$u_e = Cx\phi(t^*), \tag{2.6a}$$

where

$$t^* = Ct; \tag{2.6b}$$

C is a constant having dimensions (time)$^{-1}$; ϕ is an arbitrary function representing the nature of the unsteadiness in the external stream and has a continuous first derivative for $t^* > 0$.

Applying the similarity transformation (Gribben 1971)

$$\left.\begin{aligned} \eta = (C/\nu_w)^{\frac{1}{2}}\, Y, \quad \psi = (C\nu_w)^{\frac{1}{2}}\, x\phi(t^*)f(\eta, t^*),\\ h/h_e = g(\eta, t^*) \end{aligned}\right\} \tag{2.7}$$

to (2.5), using (2.6) and introducing

$$F = u/u_e = \partial f/\partial\eta, \tag{2.8}$$

we obtain

$$\chi\frac{\partial^2 F}{\partial\eta^2} - \frac{\partial F}{\partial t^*} + \phi\left[f\frac{\partial F}{\partial\eta} - F^2 + g\right] + \frac{1}{\phi}\frac{d\phi}{dt^*}(g - F) = 0, \tag{2.9a}$$

$$\frac{\chi}{\sigma}\frac{\partial^2 g}{\partial\eta^2} - \frac{\partial g}{\partial t^*} + \phi f\frac{\partial g}{\partial\eta} = 0. \tag{2.9b}$$

The transformed boundary conditions are

$$\left.\begin{aligned} F = 0, \quad g = g_w \quad \text{at} \quad \eta = 0\\ F \to 1, \quad g \to 1 \quad \text{as} \quad \eta \to \infty \end{aligned}\right\} \text{at time } t^*. \tag{2.10}$$

Suppose that the flow under consideration is steady at first and then changes to unsteady flow for $t^* > 0$. Then the initial conditions for $F(\eta, t^*)$ and $g(\eta, t^*)$ at $t^* = 0$ are given by the steady flow equations obtained by putting

$$\phi(t^*) = 1, \quad d\phi/dt^* = 0, \quad \partial/\partial t^* = 0 \tag{2.11}$$

in (2.9).

The quantities of chief physical interest are the skin friction and the rate of heat transfer to the surface. The equation defining the wall skin friction is

$$\tau_w = \mu_w\left(\frac{\partial u}{\partial y}\right)_w = (C\mu_w\rho_w)^{\frac{1}{2}}\, Cx\phi\left(\frac{\partial F}{\partial\eta}\right)_w = (C\mu_w\rho_w)^{\frac{1}{2}}\, u_e\left(\frac{\partial F}{\partial\eta}\right)_w. \tag{2.12}$$

The rate of heat transfer from the wall to the fluid is

$$q_w = -\frac{\mu_w}{\sigma}\left(\frac{\partial h}{\partial y}\right)_w = -(C\mu_w\rho_w)^{\frac{1}{2}}\frac{h_e}{\sigma}\left(\frac{\partial g}{\partial\eta}\right)_w. \tag{2.13}$$

Thus $(\partial F/\partial\eta)_w$ and $(\partial g/\partial\eta)_w$, the velocity and enthalpy gradients at the surface, are the critical skin-friction and heat-transfer parameters of the flow.

3. Transformation to finite co-ordinates

The system of simultaneous partial differential equations (2.9), which involves two independent variables x and t^*, both varying from 0 to ∞, is solvable for F and g with the use of implicit finite-difference formulae. In order to facilitate the application of finite-difference schemes, we transform the equations to a new system of co-ordinates wherein the indefinite limit of integration in η is replaced by a definite limit. Employing the transformation

$$\xi = 1 - e^{-\alpha\eta}, \tag{3.1}$$

where α is a constant that can be used as a scaling factor to provide an optimum distribution at nodal points across the boundary layer, and letting

$$Z = \alpha(1-\xi), \tag{3.2}$$

we arrive at the following set of equations for F and g:

$$\chi Z^2 \frac{\partial^2 F}{\partial \xi^2} + (\phi f - \chi\alpha) Z \frac{\partial F}{\partial \xi} - \frac{\partial F}{\partial t^*} + \phi(g - F^2) + \frac{1}{\phi}\frac{d\phi}{dt^*}(g - F) = 0, \tag{3.3a}$$

$$\frac{\chi Z^2}{\sigma}\frac{\partial^2 g}{\partial \xi^2} + \left[\phi f - \frac{\chi\alpha}{\sigma}\right] Z \frac{\partial g}{\partial \xi} - \frac{\partial g}{\partial t^*} = 0. \tag{3.3b}$$

The initial conditions are

$$\left.\begin{aligned} \chi Z^2 \frac{\partial^2 F}{\partial \xi^2} + (f - \chi\alpha) Z \frac{\partial F}{\partial \xi} + g - F^2 &= 0 \\ \frac{\chi Z^2}{\sigma}\frac{\partial^2 g}{\partial \xi^2} + \left(f - \frac{\chi\alpha}{\sigma}\right) Z \frac{\partial g}{\partial \xi} &= 0 \end{aligned}\right\} \quad \text{at} \quad t^* = 0, \tag{3.4}$$

while the boundary conditions are

$$\left.\begin{aligned} F = 0, \quad g = g_w \quad \text{at} \quad \xi = 0 \\ F = 1, \quad g = 1 \quad \text{at} \quad \xi = 1 \end{aligned}\right\} \quad \text{for} \quad t^* \geqslant 0. \tag{3.5}$$

Here f is given by

$$f = \int_0^\xi (F/Z)\, d\xi + f_w, \tag{3.6}$$

where

$$f_w = \begin{Bmatrix} -v_w/(Cv_w)^{\frac{1}{2}}\phi & \text{for} \quad t^* > 0, \\ -v_w/(Cv_w)^{\frac{1}{2}} & \text{for} \quad t^* = 0. \end{Bmatrix} \tag{3.7}$$

The solutions of the foregoing equations are useful for examining the velocity and enthalpy profiles at each instant t^*. Besides, they were helpful in the evaluation of the skin-friction and heat-transfer parameters, namely $(\partial F/\partial\eta)_w$ and $(\partial g/\partial\eta)_w$, at time t^* by means of the relations

$$\left(\frac{\partial F}{\partial \eta}\right)_w = \alpha\left(\frac{\partial F}{\partial \xi}\right)_w, \quad \left(\frac{\partial g}{\partial \eta}\right)_w = \alpha\left(\frac{\partial g}{\partial \xi}\right)_w. \tag{3.8}$$

By finding $(\partial F/\partial\xi)_w$ and $(\partial g/\partial\xi)_w$ at different times t^*, the response of the laminar skin friction and heat transfer at the wall to the variations in the velocity of the oncoming stream with time can be studied for different types of unsteadiness in the free-stream velocity.

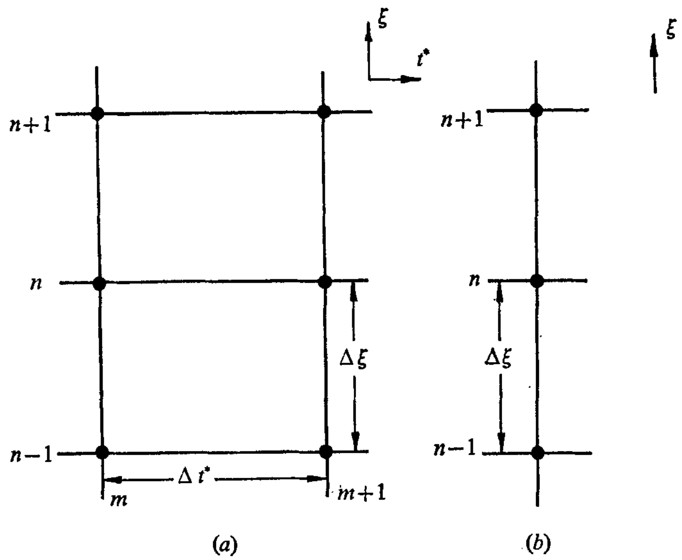

FIGURE 1. Mesh-point diagrams for Crank–Nicholson scheme. (a) Unsteady flow ($t^* > 0$). (b) Steady flow ($t^* = 0$).

4. Method of solution

A numerical solution of (3.3) subject to the initial conditions (3.4) and the boundary conditions (3.5) may be obtained by converting them into a set of implicit finite-difference equations of the Crank–Nicholson type and solving the ensuing tridiagonal matrix equations on a computer by making use of a suitable algorithm (Varga 1962; Marvin & Sheaffer 1969). The mesh-point diagram for the Crank–Nicholson scheme is shown in figure 1 (a).

Suppose that there are N nodal points in the ξ direction, so that $(N-1)\Delta\xi = 1$. Then the semi-similar boundary-layer equations (3.3) may be converted into a set of $2(N-2)$ linear algebraic equations using the following type of finite-difference equations to approximate the unknowns and their derivatives:

$$F = \tfrac{1}{2}(F_{n,m}+F_{n,m+1}), \quad F^2 = F_{n,m}F_{n,m+1}, \tag{4.1a,b}$$

$$\partial F/\partial t^* = (F_{n,m+1}-F_{n,m})/\Delta t^*, \tag{4.1c}$$

$$\partial F/\partial \xi = (4\Delta\xi)^{-1}[F_{n+1,m}-F_{n-1,m}+F_{n+1,m+1}-F_{n-1,m+1}], \tag{4.1d}$$

$$\partial^2 F/\partial \xi^2 = (2(\Delta\xi)^2)^{-1}[F_{n+1,m}-2F_{n,m}+F_{n-1,m}+F_{n+1,m+1}-2F_{n,m+1}+F_{n-1,m+1}]. \tag{4.1e}$$

The resulting equations can be conveniently written in the matrix form

$$\mathbf{A}_n\boldsymbol{\omega}_{n-1}+\mathbf{B}_n\boldsymbol{\omega}_n+\mathbf{C}_n\boldsymbol{\omega}_{n+1} = \mathbf{D}_n, \quad n = 2,3,\ldots,N-1, \tag{4.2}$$

where the vectors and the coefficient matrices are given by

$$\boldsymbol{\omega}_n = \begin{bmatrix} F \\ g \end{bmatrix}_{n,m+1}, \quad \mathbf{D}_n = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}_{n,m \text{ or } m+\frac{1}{2}}, \tag{4.3a,b}$$

$$\mathbf{A}_n = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}_{n,m \text{ or } m+\frac{1}{2}}, \quad \mathbf{B}_n = \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{bmatrix}_{n,m \text{ or } m+\frac{1}{2}}, \tag{4.3c,d}$$

$$\mathbf{C}_n = \begin{bmatrix} C_{11} & 0 \\ 0 & C_{22} \end{bmatrix}_{n,m \text{ or } m+\frac{1}{2}}. \tag{4.3e}$$

The matrix elements are given by the following equations:

$$A_{11} = \frac{\chi Z^2}{2(\Delta\xi)^2} + \frac{Z}{4\Delta\xi}(\chi\alpha - \phi f), \tag{4.4a}$$

$$B_{11} = -\frac{\chi Z^2}{(\Delta\xi)^2} - \phi F_{n,m} - \frac{1}{\Delta t^*} - \frac{1}{2\phi}\frac{d\phi}{dt^*}, \tag{4.4b}$$

$$C_{11} = -A_{11} + \frac{\chi Z^2}{(\Delta\xi)^2}, \quad B_{12} = \frac{1}{2}\left(\phi + \frac{1}{\phi}\frac{d\phi}{dt^*}\right), \tag{4.4c, d}$$

$$D_1 = -\frac{\chi Z^2}{2}\frac{\partial^2 F_{n,m}}{\partial \xi^2} + \frac{Z}{2}\frac{\partial F_{n,m}}{\partial \xi}(\chi\alpha - \phi f) - \frac{F_{n,m}}{\Delta t^*} + \frac{1}{2\phi}\frac{d\phi}{dt^*}F_{n,m} - \frac{1}{2}\left(\phi + \frac{1}{\phi}\frac{d\phi}{dt^*}\right)g_{n,m}, \tag{4.4e}$$

$$A_{22} = \frac{\chi Z^2}{\sigma 2(\Delta\xi)^2} + \frac{Z}{4\Delta\xi}\left[\frac{\chi\alpha}{\sigma} - \phi f\right], \tag{4.4f}$$

$$B_{22} = -\frac{\chi Z^2}{\sigma(\Delta\xi)^2} - \frac{1}{\Delta t^*}, \quad C_{22} = -A_{22} + \frac{\chi Z^2}{\sigma(\Delta\xi)^2}, \tag{4.4g, h}$$

$$D_2 = -\frac{\chi Z^2}{2\sigma}\frac{\partial^2 g_{n,m}}{\partial \xi^2} + \frac{Z}{2}\frac{\partial g_{n,m}}{\partial \xi}\left[\frac{\chi\alpha}{\sigma} - \phi f\right] - \frac{g_{n,m}}{\Delta t^*}. \tag{4.4i}$$

Here

$$\left.\begin{aligned} \frac{\partial F_{n,m}}{\partial \xi} &= \frac{F_{n+1,m} - F_{n-1,m}}{2\Delta\xi}, & \frac{\partial^2 F_{n,m}}{\partial \xi^2} &= \frac{F_{n+1,m} - 2F_{n,m} + F_{n-1,m}}{(\Delta\xi)^2}, \\ \frac{\partial g_{n,m}}{\partial \xi} &= \frac{g_{n+1,m} - g_{n-1,m}}{2\Delta\xi}, & \frac{\partial^2 g_{n,m}}{\partial \xi^2} &= \frac{g_{n+1,m} - 2g_{n,m} + g_{n-1,m}}{(\Delta\xi)^2}. \end{aligned}\right\} \tag{4.5}$$

The matrices $\mathbf{A}_n$, $\mathbf{B}_n$ and $\mathbf{C}_n$ and the solution vectors $\mathbf{D}_n$ are considered known and are evaluated at (n, m) or $(n, m+\frac{1}{2})$ depending on the iteration, described subsequently. The system of equations (4.2) will be complete with the values of the dependent variables at $n = 1, N$ (corresponding to the boundaries $\xi = 0, 1$ respectively) specified by the equations

$$\boldsymbol{\omega}_1 = \begin{bmatrix} 0 \\ g_w \end{bmatrix}_{m+1}, \quad \boldsymbol{\omega}_N = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{4.6}$$

Equations (4.2) together with the boundary conditions (4.6) form a set of linear algebraic equations which can be solved on a digital computer, provided that the profiles of the dependent variables at an initial time, say $t^* = 0$, are known.

An algorithm that can be used to obtain the solution $\boldsymbol{\omega}_n$ at a certain instant t^*, i.e. for a particular value of m, is (Varga 1962; Marvin & Sheaffer 1969)

$$\boldsymbol{\omega}_n = -\mathbf{E}_n \boldsymbol{\omega}_{n+1} + \mathbf{J}_n, \quad 1 \leqslant n \leqslant N-1, \tag{4.7}$$

where

$$\left.\begin{aligned} \mathbf{E}_n &= (\mathbf{B}_n - \mathbf{A}_n \mathbf{E}_{n-1})^{-1}\mathbf{C}_n, \\ \mathbf{J}_n &= (\mathbf{B}_n - \mathbf{A}_n \mathbf{E}_{n-1})^{-1}(\mathbf{D}_n - \mathbf{A}_n \mathbf{J}_{n-1}), \end{aligned}\right\} \quad 2 \leqslant n \leqslant N-1, \tag{4.8}$$

and

$$\mathbf{E}_1 = \mathbf{E}_N = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{J}_1 = \begin{bmatrix} 0 \\ g_w \end{bmatrix}_{m+1}, \quad \mathbf{J}_N = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{4.9}$$

Knowing the values of the dependent variables and appropriate derivatives at m, corresponding to time t^*, the dependent variables $\boldsymbol{\omega}_n$ at $m+1$, which corresponds to time $t^*+\Delta t^*$, can be computed by adopting the following procedure. First, the values of the matrix elements A_{11}, B_{11}, etc., are evaluated using the known values of required variables at m from (4.4) and (4.5). Next, with the help of (4.8) and (4.9) the $\mathbf{E}_n$ and $\mathbf{J}_n$ based on the values of the variables at m are calculated for all n between 1 and N, starting at the wall ($n = 1$) and proceeding to the boundary-layer edge ($n = N$). By substituting these values of $\mathbf{E}_n$ and $\mathbf{J}_n$ in (4.7) and using the boundary conditions (4.9), the values of the dependent variables $\boldsymbol{\omega}_n$ at $m+1$ are determined in the reverse order, i.e., starting from $n = N$. New values for f at $m+1$ are found on using (3.6). The derivatives of f are redetermined and the new values for f and $\boldsymbol{\omega}_n$ thus obtained are averaged over the interval with the corresponding values at m to form new matrix elements at $(n, m+\frac{1}{2})$ and the process of improving $\boldsymbol{\omega}_n$ at $m+1$ is repeated till convergence is achieved. The criterion for whether convergence has been achieved or not is whether both

$$\left|1-\left(\frac{\partial F}{\partial \xi}\right)_w^\nu \Big/ \left(\frac{\partial F}{\partial \xi}\right)_w^{\nu+1}\right| \quad \text{and} \quad \left|1-\left(\frac{\partial g}{\partial \xi}\right)_w^\nu \Big/ \left(\frac{\partial g}{\partial \xi}\right)_w^{\nu+1}\right|$$

are less than 0·0001 or not, the superscripts $\nu+1$ and ν denoting the values for the present and previous iterations. It may be noted that only the dependent terms of the solution without subscripts n, m are to be updated during the iteration process. The final values of $\boldsymbol{\omega}_n$ are again used to obtain $\boldsymbol{\omega}_n$ at $m+2$ and so on.

However, to start the solutions, the values of $\boldsymbol{\omega}_n$ and f at the initial time $t^* = 0$ have to be calculated by solving the steady equations (3.4) subject to the same boundary conditions (3.5) by a method analogous to that described above. Equations (3.4) can also be transformed into a matrix equation of the form (4.2), by making use of the grid system shown in figure 1 (*b*) and the following type of finite-difference approximation:

$$\left.\begin{aligned} F = F_n, \quad F^2 = F_n^{\nu+1}F_n^\nu, \\ \frac{\partial F}{\partial \xi} = \frac{F_{n+1}-F_{n-1}}{2\Delta\xi}, \quad \frac{\partial^2 F}{\partial \xi^2} = \frac{F_{n+1}-2F_n+F_{n-1}}{(\Delta\xi)^2}, \\ f(\partial F/\partial\xi) = f_n^\nu(\partial F/\partial\xi), \quad f(\partial g/\partial\xi) = f_n^\nu(\partial g/\partial\xi). \end{aligned}\right\} \tag{4.10}$$

The vectors and the coefficient matrices are given by

$$\boldsymbol{\omega}_n = \begin{bmatrix} F \\ g \end{bmatrix}_n, \quad \mathbf{D}_n = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{4.11 a, b}$$

$$\mathbf{A}_n = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}_n, \quad \mathbf{B}_n = \begin{bmatrix} B_{11} & 1 \\ 0 & B_{22} \end{bmatrix}, \quad \mathbf{C}_n = \begin{bmatrix} C_{11} & 0 \\ 0 & C_{22} \end{bmatrix}_n, \tag{4.11 c–e}$$

where the matrix elements are

$$A_{11} = \chi Z^2/(\Delta\xi)^2 - (f_n^\nu - \chi\alpha)\, Z/2\Delta\xi, \tag{4.12 a}$$

$$B_{11} = -2\chi Z^2/(\Delta\xi)^2 - F_n^\nu, \quad C_{11} = -A_{11} + 2\chi Z^2/(\Delta\xi)^2, \tag{4.12 b, c}$$

$$A_{22} = \chi Z^2/\sigma(\Delta\xi)^2 - [f_n^\nu - \chi\alpha/\sigma]\, Z/2\Delta\xi, \tag{4.12 d}$$

$$B_{22} = -2\chi Z^2/\sigma(\Delta\xi)^2, \quad C_{22} = -A_{22} + 2\chi Z^2/\sigma(\Delta\xi)^2. \tag{4.12 e, f}$$

The boundary conditions and the solution algorithm remain unchanged.

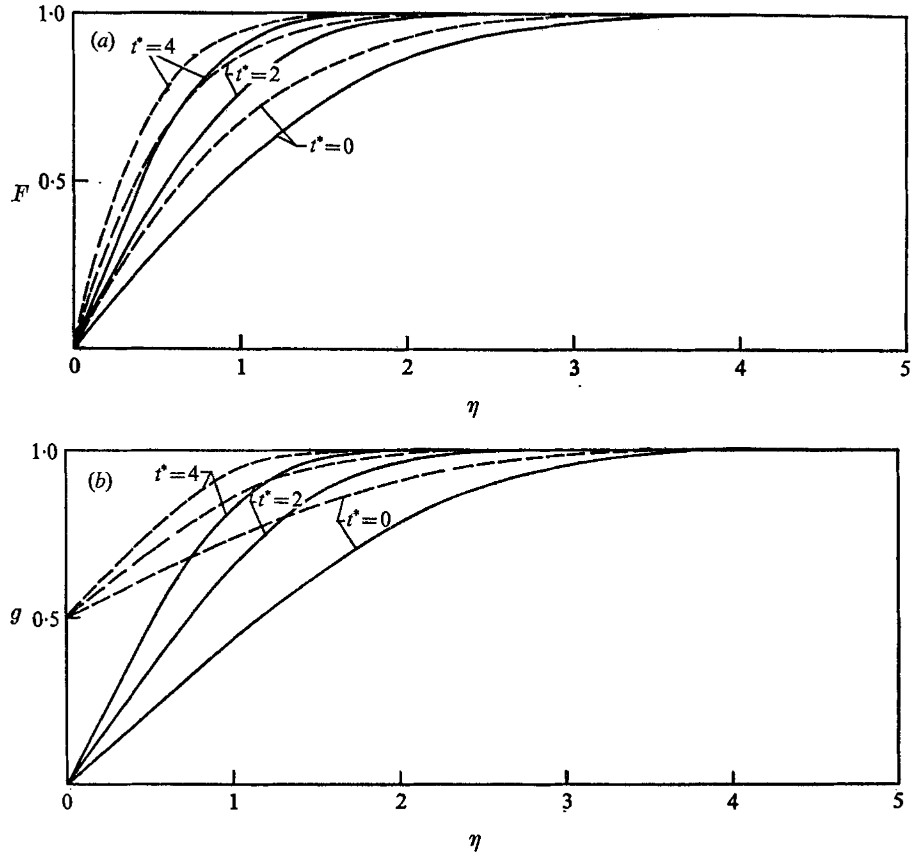

FIGURE 2. (a) Velocity and (b) enthalpy profiles for $t^* = 0$, 2 and 4. $\phi(t^*) = 1+t^*$ for $t^* \geqslant 0$. ——, $g_w = 0$; – – –, $g_w = 0{\cdot}5$.

Assuming linear profiles for the dependent variables across the boundary layer to evaluate the matrix elements from (4.12), new values for the dependent variables are obtained from (4.7)–(4.9) and (3.6). The new values thus obtained are used again in (4.12) to form new matrix elements for the next iteration and so on. The solution is iterated until the ratios

$$\left(\frac{\partial F}{\partial \xi}\right)_w^\nu \Big/ \left(\frac{\partial F}{\partial \xi}\right)_w^{\nu+1} \quad \text{and} \quad \left(\frac{\partial g}{\partial \xi}\right)_w^\nu \Big/ \left(\frac{\partial g}{\partial \xi}\right)_w^{\nu+1}$$

both differ from unity only by a quantity less than 0·0001. The final values of $\boldsymbol{\omega}_n$ and f at $t^* = 0$ are later used to determine $\boldsymbol{\omega}_n$ and f for $t^* > 0$ by the method outlined above.

5. Results and discussion

The method of solution described in the previous section has been programmed in Fortran IV and solutions have been obtained on an IBM 360 computer for two different unsteady free-stream velocity distributions characterized by

$$\phi(t^*) = 1+t^* \quad \text{and} \quad \phi(t^*) = 1+\epsilon \sin(\omega^* t^*)$$

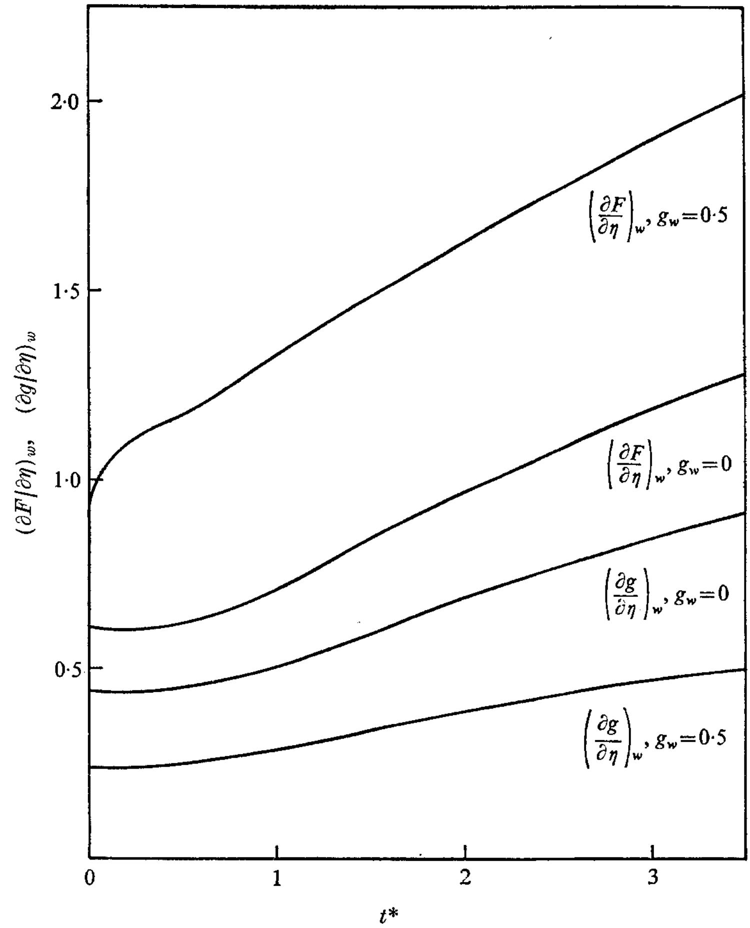

FIGURE 3. Variation of $(\partial F/\partial \eta)_w$ and $(\partial g/\partial \eta)_w$ with t^*. $\phi(t^*) = 1+t^*$ for $t^* \geqslant 0$.

(where ϵ is a small constant and ω^* is the frequency parameter). Both the skin-friction and heat-transfer parameter $(\partial F/\partial \eta)_w$ and $(\partial g/\partial \eta)_w$ have been evaluated by using (3.8). Computations have been carried out with $\chi = 1$, $\sigma = 0·72$, $\alpha = 0·5$, $\Delta\xi = 0·05$, $\Delta t^* = 0·05$ or $0·1$, $f_w = 0$ and $g_w = 0$ or $0·5$. The steady-state values thus obtained compare well with the values obtained by considering steady flow and employing quasi-linearization (Libby 1967) or any other suitable method.

(i) $\phi(t^*) = 1+t^*$

This is the case of a cylinder fixed in a stream flowing initially with a constant velocity and then, at a certain instant $t^* = 0$, starting to flow with a constant acceleration. The velocity and enthalpy profiles for $g_w = 0$ or $0·5$ at different times $t^* = 0$, 2 and 4 are shown in figures 2(*a*) and (*b*) respectively. Figure 3 depicts the variation of the skin-friction and heat-transfer parameters with time. It may be seen that, in accelerating flows, after a certain time both the skin friction and the heat transfer increase steadily with time.

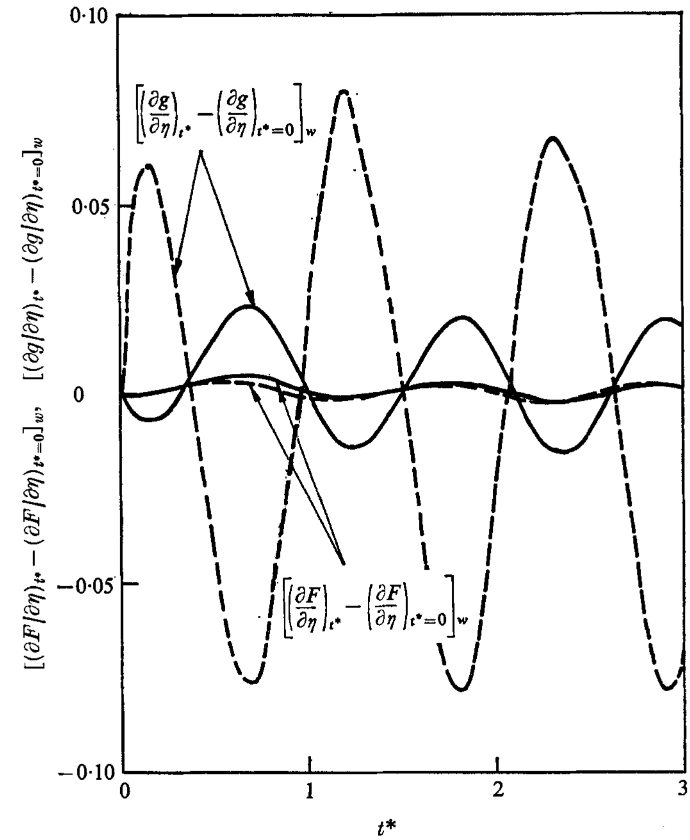

FIGURE 4. Variation of $[(\partial F/\partial\eta)_{t^*} - (\partial F/\partial\eta)_{t^*=0}]_w$ and $[(\partial g/\partial\eta)_{t^*} - (\partial g/\partial\eta)_{t^*=0}]_w$ with t^*. $\phi(t^*) = 1+\epsilon\sin\omega^*t^*$ for $t^* \geqslant 0$; $\epsilon = 0{\cdot}1$, $\omega^* = 5{\cdot}6$. ——, $g_w = 0$; – – –, $g_w = 0{\cdot}5$.

$$\text{(ii)}\quad \phi(t^*) = 1+\epsilon\sin(\omega^*t^*)$$

This example deals with a cylinder fixed in a stream fluctuating with a frequency $C\omega^*$ about a steady mean. The variation of the critical wall parameters with time is presented in figure 4 for $\epsilon = 0{\cdot}1$, $\omega^* = 5{\cdot}6$ and $g_w = 0$ or $0{\cdot}5$. It is evident from the figure that the skin-friction parameter responds more to the fluctuations in the free-stream velocity than the heat-transfer parameter. As the present solution is different from that of Gribben (1971) (the reasons were mentioned earlier), it was not possible to compare the present results for high frequency with those obtained by Gribben (1971) using the method of series expansion.

6. Concluding remarks

The flow in unsteady laminar compressible boundary layers in the immediate neighbourhood of a two-dimensional stagnation point due to an incident stream with an unsteady velocity has been investigated by applying an accurate numerical method developed by Marvin & Sheaffer (1969) for solving non-similar boundary-layer equations. The semi-similar boundary-layer equations, involving both time and the independent similarity variable, have been solved by converting them into a matrix equation with the help of approximate implicit finite-difference expressions after transforming the governing conservation equations to new co-ordinates with a finite domain. Initial values of the dependent variables required for solving the unsteady flow equations were obtained by applying the same method to the steady equations.

The method presented here could be used to study unsteady laminar boundary layers in compressible stagnation-point flow with an arbitrary free-stream velocity distribution. The method could also be employed to take into account variations with time of the density–viscosity product, Prandtl number, suction or injection at the surface or the surface temperature, provided that they were given *a priori* as functions of time.

Finally, the present study illustrates the advantages of mapping infinite intervals into finite ones and applying the appropriate implicit finite-difference schemes when handling flow problems of a complex nature, such as non-similar and semi-similar flows.

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